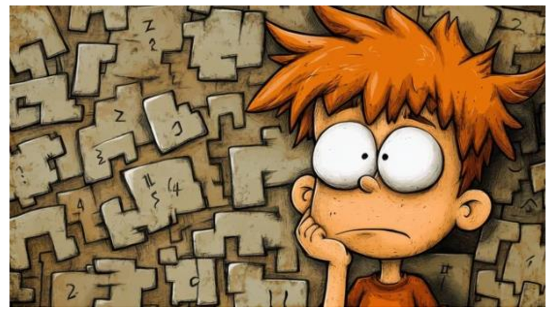
# Factoring the Sum and Difference of Cubes

We've learned that we can factor the *difference of squares* $x^2 - y^2$ into $(x + y)(x - y)$. We've also determined that the *sum of squares* $x^2 + y^2$ cannot be factored.



Now we're about to show that the ***difference of cubes*** $x^3 - y^3$ can also be factored — and perhaps surprisingly — even the ***sum of cubes*** $x^3 + y^3$ can be factored.

## ❑ *Division, Remainders, and Factors*

Is 6 a **factor** of 161? **No** — divide 161 by 6 and you'll get 26 remainder **5**. Since the remainder is not 0, 6 is not a factor of 161. In other words, 6 does not go into 161 "evenly."

Is 7 a **factor** of 161? **Yes** — divide 161 by 7 and you'll get 23, remainder **0**. Thus, 7 divides into 161 exactly 23 times. And therefore, $161 = 7 \times 23$. We have factored 161 into $7 \times 23$ by showing that the factor 7 divides into 161 without remainder.

These observations are the key to factoring the sum and difference of cubes.

## ❑ Perfect Cubes

We know that $2^3 = 8$. Since the cube of 2 is 8, we say that 8 is a ***perfect cube***. Here are some more examples of perfect cubes:

**125** is a *perfect cube* because it's the cube of 5:

$$125 = 5^3$$

**1** is a perfect cube because it's the cube of 1:

$$1 = 1^3$$

**–27** is a perfect cube because it's the cube of –3:

$$27 = (-3)^3$$

**0** is a perfect cube because it's the cube of 0:

$$0 = 0^3$$

$\boldsymbol{x^3}$ is a perfect cube because it's the cube of $x$:

$$x^3 = x^3$$

$\boldsymbol{27y^3}$ is a perfect cube because it's the cube of $3y$:

$$27y^3 = (3y)^3$$

$\boldsymbol{8n^6}$ is a perfect cube because it's the cube of $2n^2$:

$$8n^6 = (2n^2)^3$$

$\boldsymbol{64z^{12}}$ is a perfect cube because it's the cube of $4z^4$:

$$64z^{12} = (4z^4)^3$$

$\boldsymbol{(a - b)^3}$ is a perfect cube because it's the cube of $a - b$.

# Homework

**1.** a. $64m^3$ is a perfect cube because it's the cube of _____.

b. $216n^3$ is a perfect cube because it's the cube of _____.

c. $27A^6$ is a perfect cube because it's the cube of _____.

d. _____ is a perfect cube because it's the cube of $7z^2$.

e. _____ is a perfect cube because it's the cube of $-3a^3$.

## ❑ *FACTORING A SUM OF CUBES – LONG DIVISION*

We're now ready to try to factor a sum of cubes; for example, what is the factorization of

$x^3 + 8$ ? [What should we multiply to get $x^3 + 8$?]

To answer this question, we should try to divide $x^3 + 8$ by something that goes into it evenly; that is, divide $x^3 + 8$ by something that will leave a remainder of 0. But what should we divide by? Since both terms of $x^3 + 8$ are perfect cubes, let's divide it by the binomial $x + 2$, since these two terms are the cube roots of $x^3$ and 8. Maybe this will work and maybe it won't, but we've got to try something.

$$
\begin{array}{r|l}
 & x^2 - 2x + 4 \\
\hline
x+2 & x^3 + 0x^2 + 0x + 8 \\
 & \underline{x^3 + 2x^2} \\
 & \quad -2x^2 + 0x \\
 & \quad \underline{-2x^2 - 4x} \\
 & \qquad\quad 4x + 8 \\
 & \qquad\quad \underline{4x + 8} \\
 & \qquad\qquad 0
\end{array}
$$

Here's the division of $x^3 + 8$ by $x + 2$. Note that the dividend has two zeros placed in it to account for the missing terms.

Also note that the remainder is 0. This means that $x + 2$ is a factor of $x^3 + 8$ and therefore, that $x^2 - 2x + 4$ is the other factor.

Now we write the results of our long division as a multiplication problem, giving us the factorization of $x^3 + 8$:

$$\boxed{x^3 + 8 = (x+2)(x^2 - 2x + 4)}$$

Check by multiplying

## ❑ FACTORING A DIFFERENCE OF CUBES – LONG DIVISION

For our difference of cubes, let's try to factor $n^3 - 27$. What do you think one of the factors will be? Consider the binomial consisting of the individual cube roots of $n^3$ and –27, namely $n$ and – 3. This time it's your turn to carry out the long division. Here's what you should end up with:

$$\begin{array}{r|l} & n^2 + 3n + 9 \\ \cline{2-2} n-3 & n^3 + 0n^2 + 0n - 27 \end{array}$$

We now have our factorization:

$$\boxed{n^3 - 27 = (n-3)(n^2 + 3n + 9)}$$

Again, check by multiplying

<u>EXAMPLE 1:</u>

A. Factor: $N^3 - 1$

Divide $N^3 - 1$ by $N - 1$ and you should get the factorization

$$\mathbf{N^3 - 1 = (N - 1)(N^2 + N + 1)}$$

B. Factor: $8p^3 + 27$

Divide $8p^3 + 27$ by $2p + 3$. It should divide evenly, giving

$$\mathbf{8p^3 + 27 = (2p + 3)(4p^2 - 6p + 9)}$$

C. Factor: $(a + b)^3 - 125$.

This is tricky, and it will be much easier to perform the long division if we make a substitution first. If we let $x = a + b$, then the expression to factor becomes $x^3 - 125$. The appropriate quantity to divide this by would be $x - 5$. When the long division is finished, the quotient is $x^2 + 5x + 25$ with remainder 0. We therefore get the factorization

$$x^3 - 125 = (x - 5)(x^2 + 5x + 25)$$

But the original problem didn't have any $x$'s in it. So we need to substitute back the other way — when we convert each $x$ back into $a + b$, we get the factorization

$$(a + b)^3 - 125 = \big((a + b) - 5\big)\big((a + b)^2 + 5(a + b) + 25\big),$$

which can be simplified to the final answer of

$$\boldsymbol{(a + b)^3 - 125 = (a + b - 5)(a^2 + 2ab + b^2 + 5a + 5b + 25)}$$

# Homework

2. In the discussion above, we arrived at the following factorizations:

   a. $x^3 + 8 = (x + 2)(x^2 - 2x + 4)$

   b. $n^3 - 27 = (n - 3)(n^2 + 3n + 9)$

   Verify each result by simplifying the right side of the statement so that it becomes the left side.

3. Factor each expression:

   a. $x^3 - 8$ b. $n^3 + 27$ c. $z^3 + 1$

   d. $8x^3 - 27$ e. $27y^3 + 125$ f. $64a^3 - 1$

4. Factor each expression:

   a. $(x + y)^3 + 8$ b. $(a - b)^3 - 27$ c. $(p + q)^3 + 1$

5. Factor $x^5 + 1$. Hint: Divide by $x + 1$.

6. Factor $A^5 - 32$. Hint: Divide by $A - 2$.

7. Factor each expression:

   a. $w^5 + 1$ b. $c^5 - 1$ c. $y^5 - 32$

   d. $z^5 + 32$ e. $n^5 + 243$ f. $m^5 - 243$

8. Factor each expression:

   a. $x^7 - 1$ b. $y^7 + 1$ c. $u^7 - 128$ d. $z^7 + 128$

## ❑ *CREATING FORMULAS FOR FACTORING CUBES*

Factoring the sum (or difference) of cubes using long division is kind of tedious (and prone to errors). I suggest we do long division on a "generic" sum of cubes and see if we can come up with a formula that will allow us to factor the sum of cubes more efficiently.

Let's start by considering the factorization of $x^3 + y^3$. If you've understood this chapter so far, you'll understand what I'm about to do: Divide $x^3 + y^3$ by $x + y$, and confirm that the remainder comes out to be 0. Then we'll have our factorization.

$$x + y\,\overline{\big)\,x^3 + y^3}$$

We first have to pad up the dividend with some placeholders. What terms to include may not be obvious, but there's a nice pattern here: The powers on the $x$'s go down while the powers on the $y$'s go up.

$$\begin{array}{r|l} & x^2 - xy + y^2 \\ \hline x+y & x^3 + 0x^2y + 0xy^2 + y^3 \\ & \underline{x^3 + x^2y} \\ & \quad -x^2y + 0xy^2 \\ & \quad \underline{-x^2y - xy^2} \\ & \qquad\quad xy^2 + y^3 \\ & \qquad\quad \underline{xy^2 + y^3} \\ & \qquad\qquad\quad \mathbf{0} \; \checkmark \end{array}$$

Since the remainder is 0, we have our factorization:

$$\boxed{x^3 + y^3 = (x+y)(x^2 - xy + y^2)}$$

<u>EXAMPLE 2:</u> **Factor: $a^3 + 8$**

$$a^3 + 8$$

$= a^3 + 2^3$ [Here $x = a$ and $y = 2$]

$= (a+2)(a^2 - a \cdot 2 + 2^2)$ [From the formula for $x^3 + y^3$]

$= \boxed{(a+2)(a^2 - 2a + 4)}$ [Multiply out to confirm]

<u>EXAMPLE 3:</u> **Factor: $n^6 + 125w^3$**

$$n^6 + 125w^3$$

$= (n^2)^3 + (5w)^3$ [Here $x = n^2$ and $y = 5w$]

$= (n^2 + 5w)\left((n^2)^2 - n^2(5w) + (5w)^2\right)$

$= \boxed{(n^2 + 5w)(n^4 - 5n^2w + 25w^2)}$

# Homework

9. Find the factorization of $x^3 - y^3$ by dividing it by $x - y$.

10. Redo Problem #3 using our two new formulas for factoring the sum and difference of cubes.

## ❑ *To ∞ and Beyond*

**A.** Consider the expression $\boldsymbol{x^6 - y^6}$. Factor it in two ways:

1) Factor it as a difference of squares, followed by using the sum and difference of cubes.

2) Factor it as a difference of cubes, followed by a difference of squares.

Which method do you think is better, and why?

**B.** If I asked you for the *cube root* of 8, you'd probably say 2. And you're right, $2^3 = 8$. ✓ But if you have studied imaginary numbers (involving the number $\boldsymbol{i}$, defined by $\boldsymbol{i} = \sqrt{-1}$), then the fact is that 8 actually has <u>three</u> cube roots, the real root 2 we saw above, and two *non-real* roots (roots containing the imaginary number $\boldsymbol{i}$).

Here's how we can find them. Let $x$ represent a cube root of 8. This implies that $x^3 = 8$, which implies that $x^3 - 8 = 0$. We now have a cubic equation that just might produce three solutions (just like a quadratic equation (degree = 2) might yield two solutions). These three potential solutions will all be cube roots of 8. Here's where the factoring in the chapter comes in:

$$x^3 - 8 = 0$$

$$\Rightarrow \quad (x-2)(x^2+2x+4) = 0 \qquad \text{(difference of cubes)}$$

$$\Rightarrow \quad x-2=0 \text{ OR } x^2+2x+4 = 0 \qquad (AB=0 \Rightarrow A=0 \text{ OR } B=0)$$

The first equation (linear) is easy to solve: $\boldsymbol{x = 2}$, which we already knew was a cube root of 8.

To solve the second equation (quadratic) we will apply the Quadratic Formula (or we could complete the square). Since $a = 1$, $b = 2$, and $c = 4$, we can solve for $x$:

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(4)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{4-16}}{2}$$

$$= \frac{-2 \pm \sqrt{-12}}{2}$$

$$= \frac{-2 \pm 2\sqrt{3}\,i}{2} \qquad \text{(the } i \text{ is \underline{not} in the radical sign)}$$

$$= \frac{2\left(-1 \pm \sqrt{3}\,i\right)}{2} \qquad \text{(factor the numerator)}$$

$$= -1 \pm \sqrt{3}\,i \qquad \text{(divide out the 2's, and notice that we have two solutions)}$$

And we now have our three solutions to the equation $x^3 - 8 = 0$, which is equivalent to having our three cube roots of 8, as promised:

$$\boxed{2,\ -1+\sqrt{3}\,i,\ \text{and}\ -1-\sqrt{3}\,i}$$

If we allow ourselves to venture beyond the real numbers, we can find <u>three</u> cube roots of 8.

If you're up to it, verify the two complex cube roots of 8 by cubing each one and verifying that the final result will be 8 in both cases.

# Solutions

**1.** a. $4m$ b. $6n$ c. $3A^2$ d. $343z^6$ e. $-27a^9$

**2.** a. $(x + 2)(x^2 - 2x + 4) = x^3 - 2x^2 + 4x + 2x^2 - 4x + 8 = x^3 + 8$ ✓

b. You try it.

**3.** a. $(x - 2)(x^2 + 2x + 4)$ b. $(n + 3)(n^2 - 3n + 9)$

c. $(z + 1)(z^2 - z + 1)$ d. $(2x - 3)(4x^2 + 6x + 9)$

e. $(3y + 5)(9y^2 - 15y + 25)$ f. $(4a - 1)(16a^2 + 4a + 1)$

**4.** a. $(x + y + 2)(x^2 + 2xy + y^2 - 2x - 2y + 4)$

b. $(a - b - 3)(a^2 - 2ab + b^2 + 3a - 3b + 9)$

c. $(p + q + 1)(p^2 + 2pq + q^2 - p - q + 1)$

**5.** $(x + 1)(x^4 - x^3 + x^2 - x + 1)$

**6.** $(A - 2)(A^4 + 2A^3 + 4A^2 + 8A + 16)$

**7.** a. $(w + 1)(w^4 - w^3 + w^2 - w + 1)$

b. $(c - 1)(c^4 + c^3 + c^2 + c + 1)$

c. $(y - 2)(y^4 + 2y^3 + 4y^2 + 8y + 16)$

d. $(z + 2)(z^4 - 2z^3 + 4z^2 - 8z + 16)$

e. $(n + 3)(n^4 - 3n^3 + 9n^2 - 27n + 81)$

f. $(m - 3)(m^4 + 3m^3 + 9m^2 + 27m + 81)$

**8.** a. $(x-1)(x^6+x^5+x^4+x^3+x^2+x+1)$

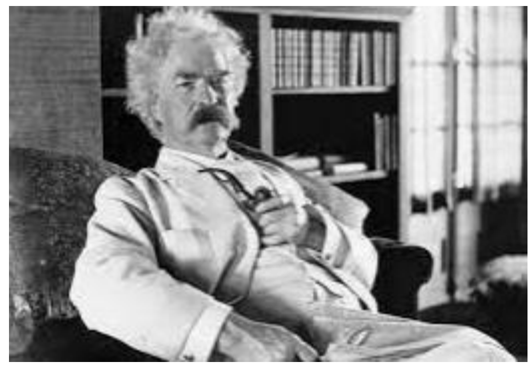
b. $(y+1)(y^6-y^5+y^4-y^3+y^2-y+1)$

c. $(u-2)(u^6+2u^5+4u^4+8u^3+16u^2+32u+64)$

d. $(z+2)(z^6-2z^5+4z^4-8z^3+16z^2-32z+64)$

**9.** $\boxed{x^3-y^3 = (x-y)(x^2+xy+y^2)}$

"Keep away from people who try to belittle your ambitions. Small people always do that, but the really great make you feel that you, too, can become great."



Mark Twain